Hamiltonian Dynamics and Three-body Problem

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Introduction

The **three-body problem**, came from the celestial mechanics, describes the motion of an object under the gravitational force of two bodies.

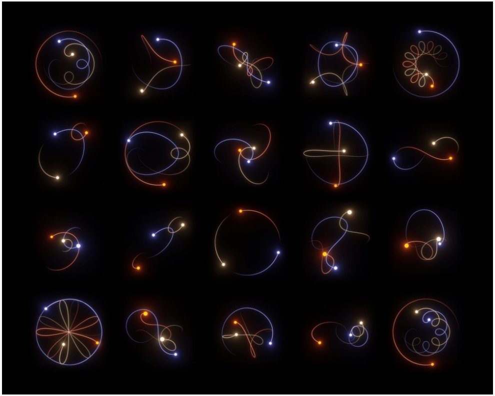


Figure 1: Various solutions of three-body problem.[1]

[1] https://www.wikiwand.com/en/articles/Three-body_problem

Contents

Main interest : Periodic orbits of three-body problem.

Contents.

- Hamiltonian dynamics
- Periodic orbits and Bifurcation
- Three-body Problem

Hamiltonian Dynamics

Classical Mechanics

Classical mechanics is governed by 2nd order ODE on $\mathbb{R}^n$ which is given by Newton's second law,

$$F = ma \quad \Leftrightarrow \quad -\nabla V(q) = \ddot{q},$$

where $q \in \mathbb{R}^n$ is the position and V is the **potential energy**.

The **mechanical energy** is defined by

$$H = K + V := \frac{|\dot{q}|^2}{2} + V(q),$$

where K is the **kinetic energy**.

Law (Classical Energy Conservation)

H is conserved along the trajectory of q determined by force F of the form $-\nabla V$ where $V = V(q)$.

Hamiltonian Mechanics

Question 1. Can we re-formulate the equation of motion via the law of energy conservation?

Consider H as a function $H = H(q,p)$ where $p \in T_q^*\mathbb{R}^n$, so

$$H(q,p) = \frac{|p|^2}{2} + V(q).$$

Then we have

$$\dot{q} = p = \frac{\partial H}{\partial p},$$
$$\dot{p} = \ddot{q} = -\nabla V = -\frac{\partial H}{\partial q}.$$

Now we have a 1st order ODE on $T^*\mathbb{R}^n$.

Hamiltonian Mechanics

Let $H : T^*\mathbb{R}^n \to \mathbb{R}$ be a function, which will be called **Hamiltonian**.

Consider the vector field X_H on $T^*\mathbb{R}^n$,

$$X_H = \begin{pmatrix} \partial_p H \\ -\partial_q H \end{pmatrix} = \sum_i \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}.$$

Theorem (Energy Conservation)

Along the trajectory of X_H, H is conserved.

We call X_H the **Hamiltonian vector field**.

Now we can impose *any* function H as the total energy of the system, and use the viewpoint of the classical mechanics.

Symplectic Structure

Question 2. How can we make X_H from H?

We naturally have a differential 1-form from H,

$$dH = \sum_i \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i.$$

If we can change (dq, dp) (T^*M) to $(-\partial_p, \partial_q)$ (TM), we get X_H.

The duality via a bilinear form $\omega = dp \wedge dq$ works by

$$\omega(X_H, \cdot) = -dH(\cdot),$$

since

$$\omega(X_H, \partial_{q_i}) = \omega\left((-\partial_{q_i} H)\, \partial_{p_i}, \partial_{q_i}\right) = -\partial_{p_i} H = -dH(\partial_{q_i}),$$
$$\omega(X_H, \partial_{p_i}) = \omega\left((\partial_{p_i} H)\, \partial_{q_i}, \partial_{p_i}\right) = -\partial_{p_i} H = -dH(\partial_{p_i}).$$

Symplectic Structure

For suitable bilinear form ω, we must have

1. **Non-degeneracy** $(\ker \omega = 0)$ for duality.
2. **Alternating property** $(\omega(X, X) = 0)$ for conservation of energy,

$$X_H(H) = dH(X_H) = -\omega(X_H, X_H) = 0.$$

3. **Closedness** $(d\omega = 0)$ for the conservation of ω,

$$\mathcal{L}_{X_H}\omega = di_{X_H}\omega + i_{X_H}d\omega = -d(dH) + i_{X_H}d\omega = i_{X_H}d\omega = 0.$$

We call any such differential 2-form ω a **symplectic form**.

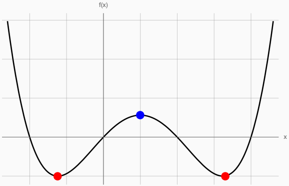
A manifold W with a symplectic form ω is called **symplectic manifold**.

The study of (W, ω) is symplectic geometry / topology / dynamics.

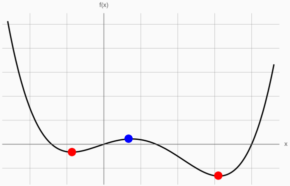
Nice introductory texts : [Arn89], [CdS01].

Periodic orbits and Bifurcation

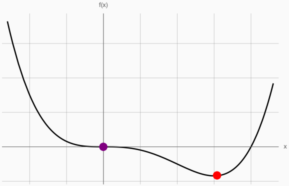
Toy Example : Function



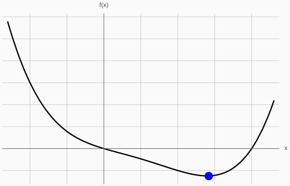
(a) $t = -1$



(b) $t = -0.5$



(c) $t = 0$



(d) $t = 1$

Figure 2: Graphs of $f_t(x) = x(x-2)(x^2+t)$

Conley-Zehnder Index

Main interest : **Periodic orbits** of a given Hamiltonian,

$\gamma : [0, \tau] \to W$ such that $\dot{\gamma}(t) = X_H(\gamma(t))$, $\gamma(0) = \gamma(\tau)$.

A periodic orbit γ is **non-degenerate** if there is no *multiplicity*.

Generically, every periodic orbit of X_H on $H^{-1}(c)$ is non-degenerate.

For non-degenerate γ, we define **Conley-Zehnder index** $\mu_{CZ}(\gamma) \in \mathbb{Z}$.

This gives grading of Floer homology, which is a **symplectic invariant**.

$\Rightarrow$ The indices of *all* periodic orbits are topologically controlled.

Bifurcation of Periodic Orbits

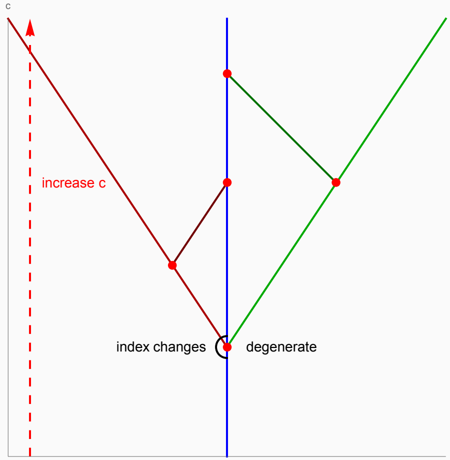


Figure 3: Bifurcation diagram

- Each point of the diagram is a periodic orbit with energy $H = c$.
- At each segment between red points, orbits are non-degenerate and the indices do not change.
- At red points, the orbit degenerates and the index changes, and families of orbits are born or vanish. We call this **bifurcation**.

Bifurcation of Periodic Orbits

Remark. Instead of changing the energy, we can perturb the Hamiltonian and see the bifurcation behavior.

$\Rightarrow$ Investigate a complicated system as a perturbation of simpler one.

Nice introductory texts : [Mil63], [Sma67], [AM78], [AD14]

Three-body Problem

Kepler Problem

The **Kepler problem** describes the motion of an object under the gravitational force of the other body.

$$E(q,p) = \frac{|p|^2}{2} - \frac{1}{|q|}.$$

In 17th century, Johannes Kepler established three laws from pure observation...

Theorem (Kepler's Laws)

1. *The Kepler orbits are conic sections with one focus at the origin. If $E < 0$, the orbits are* **ellipses***.*
2. *The areal velocity $\mathrm{dArea} = r^2\dot{\theta}$ is constant along the orbit.*
3. *The period is given by $\tau = 2\pi/(-2E)^{3/2}$.*

Setting

Let two bodies E (earth) and M (moon) has mass ratio $1 - \mu : \mu$.

$$H_t(q,p) = \frac{|p|^2}{2} - \frac{\mu}{|q - M(t)|} - \frac{1-\mu}{|q - E(t)|}$$

Assumptions

1. The third body is much lighter than M and E, so the motion of M and E are not affected by the third body.

   $\Rightarrow$ The motions of M and E are governed by Kepler problem.
2. M and E are in *circular* motion.

Circular Restricted Three-body Problem

The **circular restricted three-body problem** (CRTBP) is determined by the same $H_t(q,p)$ with

$$M(t) = (1-\mu)(\cos t, -\sin t, 0), \quad E(t) = -\mu(\cos t, -\sin t, 0).$$

We can remove the time-dependency by using *rotating frame*, which gives

$$H(q,p) = \frac{|p|^2}{2} - \frac{\mu}{|q-M|} - \frac{1-\mu}{|q-E|} + (p_1 q_2 - p_2 q_1).$$

This time-independent Hamiltonian also defines CRTBP.

Circular Restricted Three-body Problem

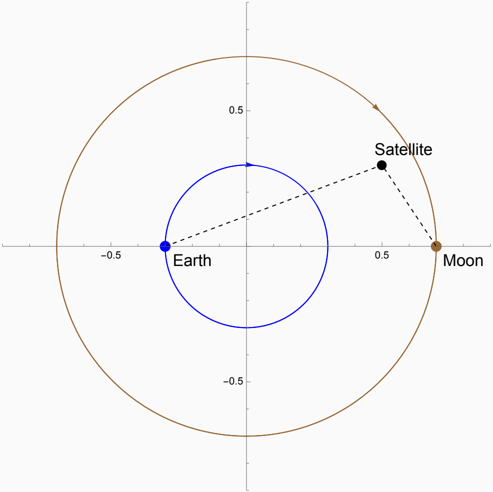


Figure 4: Illustration of CRTBP.

Hill's Region

We can rewrite the Hamiltonian as

$$\begin{aligned} H(q,p) &= \frac{1}{2}\left((p_1 - q_2)^2 + (p_2 + q_1)^2 + q_3^2\right) - \frac{\mu}{|q-M|} - \frac{1-\mu}{|q-E|} - \frac{q_1^2 + q_2^2}{2} \\ &= \frac{1}{2}|\tilde{p}|^2 + U(q). \end{aligned}$$

For the energy level c, we have $H(q,p) = c \Rightarrow U(q) \leq c$.

We call $\mathfrak{R}_c = \{q \in \mathbb{R}^3 : U(q) \leq c\}$ **Hill's region**.

Idea. For given energy level c, the position q must stay in $\mathfrak{R}_c$.

Lagrange Points

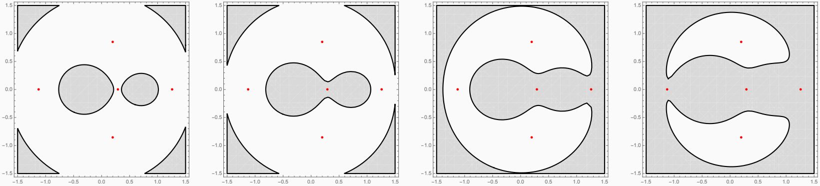


Figure 5: Hill's regions for increasing energies

The topology of the Hill's region changes through the critical values of H.

There are 5 critical points (red points), called **Lagrange points**.

At Lagrange points, $dH = 0 \Rightarrow X_H = 0$, which means that those are *equilibriums* (stationary points).

Rotating Kepler Problem

Main interest : Periodic orbits of CRTBP.

CRTBP is still hard... ⇒ Use limit cases to get ideas.

Rotating Kepler problem (RKP) is the limit case $\mu = 0$:

$$H(q,p) = \frac{|p|^2}{2} - \frac{1}{|q|} + (p_1 q_2 - p_2 q_1).$$

Note. The trajectory can be decomposed into the Kepler trajectory and the rotation along q_3-axis.

⇒ Kepler orbits invariant under the rotation are periodic orbits of RKP.

Periodic Orbits of Rotating Kepler Problem

There are 4 non-degenerate periodic orbits of RKP for generic energy.

- Planar circular orbits : **retrograde orbit** γ_+ and **direct orbit** γ_-.
- **Vertical collision orbits** : north and south $\gamma_{c_\pm}$.

Theorem ([Lee25], ArXiv preprint)

Classification of periodic orbits of RKP and computation of indices.

Periodic Orbits of Rotating Kepler Problem

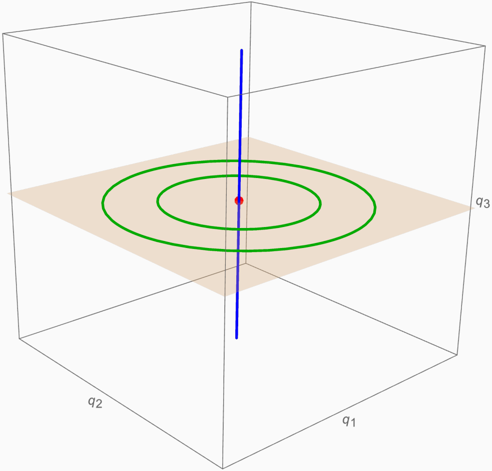


Figure 6: Non-degenerate orbits of RKP.

Periodic Orbits of Circular Restricted Three-body Problem

- We can consider CRTBP as a perturbation of RKP, and consider the family of non-degenerate orbits.
- 4 non-degenerate orbits of RKP also exist in CRTBP, which can be shown numerically, but not analytically.

For example, **retrograde orbit** is important because it's expected to :

- bifurcated from the retrograde orbit of RKP.
- have the smallest period.
- non-degenerate under the first critical energy.

Analytically, there is a candidate obtained by *Birkhoff shooting method*.

Birkhoff Shooting Method

Theorem (Birkhoff)

For $0 < \mu < 1$ *and* $c < H(\ell_1)$, *there exists a solution of CRTBP* $(q_1, q_2) : [0, \tau] \to \mathbb{R} \times (-\infty, 0]$ *such that*

1. $q_2(0) = q_2(\tau) = 0$.
2. $q_1'(0) = q_1'(\tau) = 0$.
3. $\ell_3 < q_1(0) < -\mu < \ell_1$.

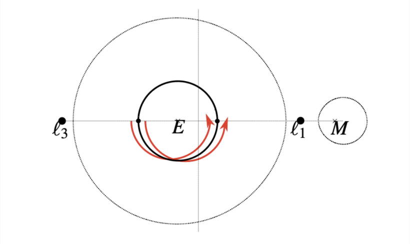


Figure 7: Birkhoff Shooting Method (courtesy of Otto van Koert)

Final Remarks

- Many numerical results about CRTBP are known, but without the aid of numerics, only little is known.
- Some simpler limit cases of CRTBP (RKP, Euler problem, Hill's lunar problem) can be studied.
- Still, CRTBP presents many interesting questions, both mathematically and practically.

Nice introductory texts : [FvK18], [Cel10]

Thank you for your attention!

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